



Teaching linear equations: Case studies from Finland, Flanders and Hungary

Paul Andrews^{a,*}, Judy Sayers^{b,1}

^a Faculty of Education, University of Cambridge, 184 Hills Road, Cambridge CB2 8PQ, UK

^b School of Education, The University of Northampton, Boughton Green Road, Northampton NN2 7AL, UK

ARTICLE INFO

Keywords:

Comparative mathematics research
Linear equations
Finland
Hungary
Flanders
Mathematical embodiments

ABSTRACT

In this paper we compare how three teachers, one from each of Finland, Flanders and Hungary, introduce linear equations to grade 8 students. Five successive lessons were video-taped and analysed qualitatively to determine how teachers, each of whom was defined against local criteria as effective, addressed various literature-derived equations-related problems. The analyses showed all four sequences passing through four phases that we have called definition, activation, exposition and consolidation. However, within each phase were similarities and differences. For example, all three constructed their exposition around algebraic equations and, in so doing, addressed concerns relating to students' procedural perspectives on the equals sign. All three teachers invoked the balance as an embodiment for teaching solution strategies to algebraic equations, confident that the failure of intuitive strategies necessitated a didactical intervention. Major differences lay in the extent to which the balance was sustained and teachers' variable use of realistic word problems.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

Learning to solve a linear equation in a single unknown (from here on referred to as a linear equation) is a key component of school mathematics. As a topic it stands on the border between mathematics as concrete and inductive and mathematics as abstract and deductive. It offers, for most learners, one of the first authentic opportunities for them to connect their understanding of arithmetic and the symbolism of mathematics. It is one of the first genuine opportunity students receive to apply mathematics as a tool for managing whole classes of mathematical and real-world problems. Significantly, it is a topic common to all the world's lower secondary curricula (Schmidt et al., 2001). In this paper we examine how three case study teachers, one from each of Finland, Flanders and Hungary, present this iconic topic before discussing their presentations against the literature concerning students' equations-related difficulties.

2. Issues in the learning of linear equations

2.1. Student difficulties with the equals sign

It is probably unsurprising to find the curricular transition from arithmetic to algebra as the cause of many equations-related difficulties (Kieran, 2006). It is problematic due, not least, to ambiguities with regard to the role and meaning of the

* Corresponding author. Tel.: +44 1223 767569.

E-mail addresses: pra23@cam.ac.uk (P. Andrews), judy.sayers@northampton.ac.uk (J. Sayers).

¹ Tel.: +44 1604 892074.

symbols of mathematics in general and the equals sign in particular (Freudenthal, 1983). With regard to the equals sign, expressions like $4 + 3$ can be construed either arithmetically or algebraically. Arithmetically it is a command to execute an operation, reflecting *procedural* (Kieran, 1992), *operational* (McNeil & Alibali, 2005; Sfard, 1995) or *canonical* (Sherman & Bisanz, 2009) expectations. Algebraically it is an object on which other operations may be performed, reflecting *relational* (Kieran, 2006), *structural* (Sfard, 1995; McNeil & Alibali, 2005) or *noncanonical* (Sherman & Bisanz, 2009) expectations. Successful equation solving requires a relational rather than operational understanding (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Knuth, Stephens, McNeil, & Alibali, 2006). Unfortunately, many students' early experiences of arithmetic reinforce the belief that the equals sign is a command to operate rather than an assertion of equality between two expressions, a problem exacerbated by the unfortunate collusion of teachers and textbooks (Baroody & Ginsburg, 1983; MacGregor & Stacey, 1997; Sherman & Bisanz, 2009). For example, when faced with equations of the form $8 + 4 = x + 5$, many students apply additive operations to any or all of the three digits in accordance with their conception of the equals sign as an instruction to operate (Alibali, 1999; Falkner, Levi, & Carpenter, 1999).

The development of a relational understanding of the equals sign, however, is not just a function of the ways in which teachers and texts present arithmetic. There is evidence that it is a function of children's cognitive maturity (Baroody & Ginsburg, 1983; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011) and occurs at different ages in different students (Alibali et al., 2007). Others, however, have argued that an operational conception persists throughout the school years (Knuth et al., 2006) and even into early adulthood (McNeil & Alibali, 2005). Such problem may be exacerbated by teachers not only underestimating the proportion of students who hold a relational conception of the equals sign but also failing to understand how the mathematical performance of students holding an operational conception is compromised by it (Asquith, Stephens, Knuth, & Alibali, 2007). That said, intervention studies have shown that students' conceptions of the equals sign can be changed, although their understanding may still fall short of that required of a mathematician (Baroody & Ginsburg, 1983). Unsurprisingly, when one considers the importance of a relational understanding, conceptually focused interventions have been more successful than procedurally focused interventions (Rittle-Johnson & Alibali, 1999), particularly when allied to appropriate "classroom discussions on the children's progressive interpretations of the equal symbol" (Saenz-Ludlow & Walgamuth, 1998, p. 185). In this respect, the teacher's role is particularly important because, even when textbooks encourage relational equivalence, without teachers explicitly drawing attention to it little change will emerge (Rittle-Johnson et al., 2011).

2.2. Distinguishing between arithmetical equations and algebraic equations

A related problem lies with the distinction between arithmetical and algebraic equations. Arithmetical equations, with the unknown in one expression only, are susceptible to undoing (Filloy & Rojano, 1989) or knowledge of arithmetic (Herscovics & Linchevski, 1994). However, algebraic equations, with unknowns on both sides, cannot be solved by arithmetic-based approaches and require not only that the learner "understand that the expressions on both sides of the equals sign are of the same nature" (Filloy & Rojano, 1989, 19) but also that they are able to operate on the unknown as an entity and not a number. In this manner, arithmetical equations are procedural, in that any undoing of an equation like $3x + 1 = 10$ requires only operations on numbers, while algebraic or non-arithmetic equations are structural in that equations like $3x + 1 = 11 - x$ require operations on algebraic entities (Boulton-Lewis et al., 1997; Kieran, 1992). For a variety of reasons, particularly the teaching approaches and text books used by teachers, many students fail to develop the pre-algebraic ideas necessary for successful transition as they are "reduced to performing meaningless operations on symbols they do not understand" (Herscovics & Linchevski, 1994, 60), a failure they describe as a *cognitive gap*. Filloy and Rojano (1989), on the other hand, argue that learners' failure is less a consequence of a cognitive gap but a consequence of the shift from arithmetic to algebraic and reflects a *didactic cut*. Pirie and Martin (1997), however, seemingly more aligned with notions of cognitive gap, dismiss the didactic cut as a myth, arguing that "this implied specific cognitive difficulty, is, in reality a notion imposed by the observer, with hind sight, to explain an artefact of particular methods of teaching" (161).

2.3. Student management of equations-related word problems

Translating words into equations is not only important in many areas of mathematics but, in its involving the identification of relevant information, matching key words with corresponding algebraic symbols and constructing a relationship between them, is difficult (Pawley, Ayres, Cooper, & Sweller, 2005). Students experience various problems due to "syntactic and semantic factors inherent in the language of mathematics" (Mestre, 1988, p. 206). With respect to syntactic difficulties, statements like *a number added to six equals nine*, with the unknown appearing near the beginning of the statement, have a clear syntactic structure that makes them amenable to a sequential left to right approach that is typically determined by word order and known generally as a *syntactic translation* (MacGregor & Stacey, 1993). However, statements like *in ten years Sara will be sixteen years old*, where the unknown has to be inferred, are generally irresolvable by means of syntactic translation. With respect to semantic difficulties, statements like *six times a number is equal to a second number* allude to a different problem whereby, failing to acknowledge that n cannot be in both terms or thinking that 2 represents the second number, students write equations of the form $6n = n$ or $6x = 2$ (Mestre, 1988). A well-known example, frequently invoking what is known as a variable reversal, is the equation $6s = p$ produced by students to represent the statement *there are six times as many students as professors*.

Mestre concludes that most problems are combinations of the two forms of difficulty and that where the syntax allows for a *sequential left to right* approach students are likely to construct a correct equation. Otherwise variable reversal is likely to emerge. Indeed, variable reversal appears to be both widespread and long lasting. Pawley et al. (2005) examined grade 8 students' equation formation and found that around 90% of the errors made, in two separate studies, were due to variable reversal. Wollman (1983), in a study of female teacher education students, found that when asked to translate statements into equations, including the professor and student problem, around half his cohort made reversal errors, although the majority of these were able to self-correct when prompted. As with their understanding of the equals sign, students' success on translating text into equations is frequently compromised by text books that "teach students to form number sentences or algebraic equations by matching specific verbal cues to mathematical symbols from left to right" (MacGregor & Stacey, 1993, p. 218).

In respect of word problems and the extraction of linear equations, Stacey and MacGregor (2000) found, even when exhorted to use algebra, students writing formulae that would yield the solution from the information given rather than an equation to be solved and, in so doing, present their reasoning as a process of undoing. Other studies have yielded similar findings. For example, most sixth grade students can "write equations using variables, albeit typically in nonstandard form as lists of operations to be 'undone'" (Swafford & Langrall, 2000, p. 107). Similarly, Humberstone and Reeve (2008) found, in respect of 12-year-old girls, only a small proportion being able to interpret in relational rather than operational ways the word problems given them. The majority remained located in an operational mode "consistent with a sequence of transition states from an arithmetic-bound understanding of mathematical expressions to algebraic competence" (p. 366). Finally, the majority of Capraro and Joffrion's (2006) 668 grade 7 and 8 students, even in a multiple choice context, were unable to convert word problems into arithmetical equations, prompting a conclusion that most were neither procedurally nor conceptually ready to translate the written word to an equation.

2.4. Checking the solution

Lastly, once completed, a solution is typically checked against the original equation, something students find difficult. Typically, these difficulties fall into two forms (Perrenet & Wolters, 1994). The first, a pattern of executional misconception, occurs when students conclude they have demonstrated the desired identity before completing all the steps of the checking procedure. In general, they understand the checking procedure, but due to inattention, omit steps, which can easily be detected and corrected. The second, a pattern of structural misconception, is more serious and is caused by uncertainty and wishful thinking. In such instances, identifying and correcting errors is difficult. Students are aware of the goal they wish to attain – an identity – but are typically unaware as to how to get there. Interestingly, Pawley et al. (2005) found that teaching students to check a solution contemporaneously with equation-related instruction was counter-productive. However, delayed instruction on checking was shown to benefit all participants, both low and high achieving.

3. Research on approaches to the teaching of linear equations

In general, introductory teaching of linear equations takes one of two forms. The first, privileging the syntax of mathematics, presents equations as mathematical entities entirely located within a world of mathematics, while the second privileges the exploitation of real world contexts as a way of providing meaning (Filloy & Rojano, 1989; Pirie & Martin, 1997). Syntactic approaches are typically characterised either by the transposition of terms from one side of the equation to the other or by operating on both sides with additive or multiplicative inverses although Pirie and Martin (1997, p. 161) add that the reality for most students is a progression "from simple equations of the form $x + b = d$ " through several intermediate forms before "finally encountering $ax + b = cx + d$ ". In terms of real world contexts, Pirie and Martin (1997, p. 163) comment that despite the "intention of giving meaning" the contexts used by teachers either reflect an adult's world or "insult the intelligence of secondary pupils with cute little animals and cartoon figures, more appropriate to the development of primary children".

A number of studies have focused on how different embodiments or representations have been used in the teaching of linear equations. At a general level it is argued that they allow new concepts, entities and operations to become endowed with meaning (Da Rocha Falcão, 1995; Filloy & Rojano, 1989) and can facilitate the link between concrete and abstract thinking (Brown, Eade, & Wilson, 1999; Linchevski & Williams, 1999). However, while concrete models may act as analogues for the intended abstractions (English & Sharry, 1996; Warren & Cooper, 2005), they may mask teachers' intended learning outcomes, necessitating appropriate teacher interventions to ensure appropriate abstraction (Filloy & Sutherland, 1996).

At the particular level of linear equations, a variety of embodiments exist. For example, according to Wijers (2001), the cover-up method is frequently exploited by Dutch teachers. An approach typical in Singapore (Fong & Chong, 1995), presents each expression in an equation as a set of rectangles laid on top of each other. For example, $2x + 10 = 4x + 2$ would be represented as in Fig. 1, from which it is alleged to be straightforward for students to infer that $2x = 8$.

x	x	10		
x	x	x	x	2

Fig. 1. Fong and Chong's (1995) representation of $2x + 10 = 4x + 2$.

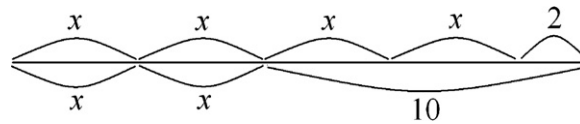


Fig. 2. Dickinson and Eade's (2004) representation of $2x + 10 = 4x + 2$.

A related embodiment (Dickinson & Eade, 2004) exploits the number line to create a model isomorphic to that of Fong and Chong (1995), as shown in Fig. 2.

However, these representation are not well-discussed in the literature and, apart from a brief allusion to the cover-up method by Sami, the Finnish teacher, none was seen in case study lessons. The most commonly discussed embodiment, and the one observed in all three case study sequences, seems to be the balance. Its advocates argue that it “offers a support for symbolic representation, which semantically and syntactically sets the foundations for the introduction of algebraic formalisms” (Da Rocha Falcão, 1995, p. 72). It “also copes with the need to attend to the equation as an entity rather than an instruction to achieve a result” (Warren & Cooper, 2005, p. 60). Its critics argue that it cannot represent negatives in anything but a contrived way (Nogueira de Lima & Tall, 2008; Pirie & Martin, 1997) and is an inappropriate embodiment due to its being unfamiliar to students used to electronic scales (Pirie & Martin, 1997), an argument countered by Da Rocha Falcão's (1995, p. 80) description of the balance as “a culturally familiar artefact”.

Despite such criticisms a number of intervention studies, typically undertaken with lower secondary students, have examined the impact of the balance on learning. Warren and Cooper (2005), mindful of the inadequacies of the balance with respect to negative quantities, concluded, that it helped “children in approaching problems with unknowns, particularly solving problems with unknowns on both sides of the equal sign” (70). Furthermore, it not only provided a vocabulary with which children could articulate their thinking but also facilitated their seeing the equals sign as representing equivalence rather than an instruction to calculate. Vlassis (2002), having introduced her students to arithmetical equations, presented them with a word problem from which an algebraic equation emerged. This equation prompted the introduction of the balance and the principles of equation solving by transformation. At the end of her intervention she found that most students had assimilated the conceptual basis of the balance and of the need to do the same thing to both sides. In Araya et al.'s (2010) experimental intervention randomly assigned students watched 15 min video-recordings on either traditional approaches based on transposition or an analogical balance scale approach with boxes of candies representing the unknown. Against a range of measures and problem types, the analogue group outperformed significantly the traditional group.

Boulton-Lewis et al. (1997) represented the balance by means of the physical laying of a stick between two sets of equal objects, unknowns by paper cups and integers by counters. Post-intervention evaluations found students preferring to use intuitive approaches rather than the concrete representations taught them. However, not only were students only ever asked to solve arithmetical equations, equations they could solve by means of arithmetical procedures, but also the balance, itself an embodiment of an equation, was subjected to an additional embodiment in the form of a stick. Other studies have also highlighted the extent to which equations-related teaching does not always achieve what might reasonably be expected. Steinberg, Sleeman, and Ktorza (1991) examined the extent to which students, all of whom had been taught to solve equations by adding the same expressions to both sides, were able to recognise equivalent pairs of equations. They found that while some students adopted approaches commensurate with an awareness of equivalence many identified equation pairs correctly through a process of solving and comparing. They concluded that “students' need to compute the solutions to the equations probably suggests that they were not sure that a simple transformation yields an equation with the same solution” (117). In similar vein, Nogueira de Lima and Tall's (2008, 9) interview study set out to examine “students' conceptions of equations, what solving methods they use and what previous experiences interfere in their work with equations”. They found not a single student invoking a general principle to warrant his or her reasoning, with all indicating the fragility of their knowledge by asserting that equation solving involved “simply moving symbols around” (Nogueira de Lima & Tall, 2008, p. 10). Thus, it seems that intervention studies have not always been successful.

Interestingly, by way of warranting this study, despite the plethora of equations-related research, little has focused on what teachers actually do in their classrooms. There are exceptions, and Pirie and Martin's study of Alwyn is one example. However, Alwyn's exploitation of arithmagons was idiosyncratic and certainly not one represented in the wider literature. In this paper, therefore, we consider, in depth, how locally defined effective case study teachers from Finland, Flanders and Hungary present this important topic to their students. In so doing, we pay particular attention to the ways in which the equations-related issues and problems discussed above are addressed.

4. Method

This paper draws on data from the European Union-funded, Mathematics Education Traditions of Europe (METE) project. Based in Cambridge, England, the project examined aspects of mathematics teaching in Flanders,² England, Finland, Hungary

² From the perspective of the METE study, Flanders, the autonomous Dutch-speaking community of Belgium, is construed as a country. This distinction is well known in the literature with, for example, Flanders being reported as a different educational system from Wallonia, the French-speaking community of Belgium in all the TIMSS studies in which it has participated.

and Spain and drew extensively on video recordings of sequences of lessons taught by teachers defined locally as competent in the manner of the learner's perspective study (Clarke, 2006). Thus, while it is not possible to comment on the attainment of project students, it would be reasonable to assume that project teachers would be as successful as any of their national peers. Indeed, all three teachers worked in partnership with their local universities as mentors in initial teacher training. In the particular case of Sami, the Finnish teacher, this would have required his being able "to prove (that he is) competent to work with trainee teachers" (Sahlberg, 2011, p. 36). Pauline, in Flanders, had also been involved in a government-funded teacher development project involving the production of video-taped lessons of exemplary teaching for use in teacher education. Emese, in Hungary, was also gave talks to larger groups of students as part of the university-based course. Thus, against different but locally important criteria, each teacher could be construed as reflecting systemic perspectives on effective practice.

Four sequences of five lessons were filmed on the same topic in each country. One of the topics was linear equations, taught to grade 8 students, chosen because of its importance in the transition from arithmetic to algebra. Videographers focused on the teacher whenever he or she was speaking. Teachers wore radio-microphones while a static microphone was placed strategically to capture as much student talk as possible. For each sequence, according to agreed project procedures, the first two lessons were transcribed and translated by English speaking colleagues working in the project universities. The translations, which were verified by Finnish-, Hungarian- and Dutch-speaking graduate students in mathematics education, enabled the creation of subtitles to facilitate a foreigner's viewing and comprehension of the lessons.

It is important to acknowledge that the data on which this paper is based were collected by local researchers familiar with the teachers' contexts. The analyses presented here represent our attempts, as cultural outsiders, to understand how participants enact their roles in classrooms culturally different from those with which we are familiar. In such circumstances, where researchers are cultural outsiders, there is a danger of inaccurate reporting due to incomplete understanding of the culturally embedded issues underpinning participants' actions (Liamputtong, 2010). With this in mind, our analyses were discussed with and their accuracy confirmed by the same graduate students who verified the translated transcripts. Consequently, while mindful that the results of case study are never intended to be generalisable, we are confident that our interpretations resonate closely with our graduate students' experiences of mathematics teaching in their respective countries.

Each video, with and without subtitles, was viewed several times to get a feel for how the lesson played out. Next, two complementary documents were produced for each lesson. The first was a lesson narrative in which participants' actions and utterances were recorded in as much detail, and with as little interpretation, as possible. The purpose, initially, was to construct a description of the lesson. These narratives were read against repeated viewings and increasingly refined. With each viewing, a better developed, but still tentative, understanding emerged with respect to the key elements of each teacher's conceptualisation and presentation of linear equations. Significantly, the lesson narrative was constructed with no explicit reference to the written transcript; it represented what we had inferred only from repeated viewings of the subtitled videos. The second document was an annotated transcript. With each viewing, the episodes of the lesson were identified and notes made on the transcript as to our emergent understanding of the discourse being played out, the roles of the participants and initial theorising as to the nature of the activities being seen. As with the narrative, the annotated transcript was refined with each viewing of the video. For those lessons with no transcripts, shortened narratives were also produced as supplementary data on which we were able to draw. By way of triangulation, for each lesson, one of us wrote the narrative and the other the annotated transcript. For the lesson following, we swapped roles. Thus, for each lesson, two perspectives were brought to bear, and it is on these data that the following is based.

5. Results

As we viewed and reviewed the lessons, read and reread the available transcripts and lesson narratives, we became aware that all three sequences comprised four phases that we have come to call, definition, activation, exposition and consolidation. In general, the definition phase, which was always a whole-class activity, introduced students to the notion of an equation and, either implicitly or explicitly, presented a definition. The activation phase, which was predominately whole-class, exploited intuitive approaches, typically presented orally and solved mentally, to the solution of arithmetic equations. The exposition phase, which was always whole class, exposed the inadequacies of intuitive approaches and warranted the introduction of the balance through an initial presentation of an algebraic equation. Lastly, the consolidation phase, which incorporated both whole-class and individual working, enabled students to practice and exploit their newly acquired skills. It is against these four phases that we structure the presentation of our findings.

5.1. The definition phase

Lasting up to one lesson, the definition phase saw Sami in Finland, Emese in Hungary and Pauline in Flanders introducing and defining what is meant by an equation. The presentation of the definition varied considerably from Sami's explicit and formally presented definition to Emese's implicit definition.

5.1.1. Sami's definition phase

The first Finnish lesson found Sami, having discussed the results of a test undertaken the previous lesson, declaring and writing simultaneously that an equation was *two expressions denoted as being of equal magnitude*. He invited his students to

Marge's age	34	35	36
Children's total age	12	15	18
Difference	22	20	18

$\begin{array}{c} +1 \quad +1 \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ 34 \quad 35 \quad 36 \\ \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \\ +3 \quad +3 \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ 12 \quad 15 \quad 18 \\ \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \\ -2 \quad -2 \end{array}$

Fig. 3. Pauline's table of values for the Simpsons' ages problem.

think about the meaning of his definition before writing six “sentences”, as he called them: $5, x - 1, x = 3, 5 + 3 = 7, 3x - 1 = 4, x^2 = 8$, and asking students to decide which were equations and which were not. Through constant reference to the definition, the ‘sentences’ were categorised. For example, with respect to third one, Sami asked, *what about problem c, x equals 3, Olli?* Olli replied, yes, with Sami commenting, *yes, there's a sign of equality and expressions exist on both sides. So, c is an equation.* Those accepted as equations were then discussed from the perspective of truth and a classification emerged that equations could be conditionally true, always true or always false. Throughout this process Sami managed the discourse by asking closed questions like: *Is the sentence an equation? Or Is the equation true or false?* In terms of student responses he tended to repeat what was said by the respondent.

5.1.2. Emese's definition phase

Emese began her first lesson by offering her class, orally, several open sentences to revisit the role of the basic set in determining a statement's validity. These included the following, a horse has four legs, the longest river in Hungary is the Drava, $7 + 3 = 10$, $5 + x > 6$, the tallest peak in the Matra mountains is, and 15 is divisible by 4. The whole class discussed each statement from the perspective of its truth or otherwise, and concluded that some were true, others false and some open, as in $5 + x > 6$. At this point she introduced the *basic set* and suggested, initially, that it should be the set of all mountain peaks. She asked the class, *what is the solution of the first open sentence, $5 + x > 6$?* Several students respond in unison, *there is no solution.* She continued by asking *what would be the case for the longest river if the basic set were the positive integers*, to which her students again replied, *there is no solution.* This continued with her asking students to consider the statements against different basic sets before she initiated a discussion whereby an equation was defined, and written, as comprising two expressions connected by an equals sign. She asserted, through her questioning, that equations may or may not contain variables or unknowns depending on circumstances and that they were always true, sometimes true or never true. Finally she operationalised her definition through an exercise in which three sentences, $5 - x = 8$; $5 - x > 6$ and $x \cdot 2 = 7$, were written on the board. Students were given 3 min to find solutions in relation to the basic set $-3 \leq x \leq 3$ before solutions were shared. In each case a different student called out the set of values that would make the statement true.

5.1.3. Pauline's definition phase

Much of the first lesson was based around a single problem involving characters from the cartoon, the Simpsons. After students had contributed the names of the family, they were told that the mother, Marge, is currently 34 years old while the children, Bart, Lisa and Maggie, are 7, 5 and 0 years old respectively, which Pauline wrote on the board. They were then asked, in how many years' time would the sum of the three children's ages equal their mother's? Pauline structured her students' working by drawing a table of values, as shown in Fig. 3, on the board before completing collaboratively the first three columns. That is, volunteer students contributed the sum of the children's ages and the difference in each case. In so doing, she asked what they noticed with regard to the numbers in each row and, in response to student suggestions, annotated in red the relationships. Finally, she invited them to complete the task individually.

After 5 min of working, during which Pauline circulated the room checking and responding to individual queries, she paused and repeated rhetorically the question, *after how many years will the sum of the children's ages be equal to that of their mother?* After another minute, the question was repeated and the answer 11 years was agreed. At this point, Pauline did not complete the whole table but the final column with the figures, 45, 45 and 0. Then she discussed with her students how the final figure on the top row was derived by adding 11 times 1 to the 34, how the middle figure of 45 derived from adding 11 times 2 to the 12 and, finally, how the zero was the result of adding 11 times -2 to the 22.

Following this, Pauline initiated a lengthy discussion, including her writing the salient points in detail, on the notion of an unknown as a means of representing the number of years that would pass before the two sums would be equal. This led to her writing on the board that if x represented the number of years necessary for the two sums to be equal then Marge would acquire $34 + x$ years, while the children, each discussed in turn, would reach $7 + x$, $5 + x$ and x respectively. This led to $34 + x = 7 + x + 5 + x + x$ being written on the board before being simplified to $34 + x = 12 + 3x$. At this stage, Pauline wrote the heading, *equations*, on the board. Finally, in this episode, and exploiting an overhead projector, Pauline demonstrated, by means of a previously prepared acetate and with no student input, how each of the two rows of the table of values could be represented graphically to show an intersection after 11 years.

5.1.4. Summary

In summary, Sami and Emese presented explicit definitions, although Sami's, presented both orally and in writing, involved no student input. In essence, both their definitions and tripartite classification were operationalised collectively through analyses of various sentences. However, while Sami's were located entirely within a world of algebraic formalism, Emese grounded her presentation in a variety of contexts to highlight the importance of the basic set. Moreover, her use of inequalities provided a more general entry to equations and equation solving. Pauline exploited a *realistically* derived equation to define, implicitly, both equation and equation solving. The latter was undertaken implicitly by reference to both the table and graphs. She made no allusion to equation types nor made any explicit attempt to explicitly solve the algebraic equation she had derived. Interestingly, the manner by Sami and Emese defined and operationalised equations, particularly with regard to their explicit emphases on two expressions being separated by an equals sign, appeared to address students' construing the equals sign as a command to operate. Both Emese and Pauline, in their deployment of different forms of realistic contexts, attempted to ground the work in their students' world and, we argue, in so doing did not "insult the intelligence of secondary pupils with cute little animals and cartoon figures, more appropriate to the development of primary children" (Pirie & Martin, 1997, p. 163)

5.2. The activation phase

The activation phase saw all three teachers, as preparation for their main presentations, revisit material covered earlier in their students' careers to both contextualise and facilitate the material that followed. In general, this incorporated teachers offering, in different guises, arithmetical equations and encouraging their students to see that solutions can be found by recourse to a common-sense application of arithmetical operations.

5.2.1. Sami's activation phase

Sami began his second lesson by asking for a conditional equation. One student suggested $x + 5 = 2$, with a second offering -3 as a solution. Sami's response was to comment, *well, this kind of an equation is easy to solve. And, if you can solve it mentally, just write down what x should be. But we cannot be this lucky every time, take this for example.* He then wrote $x/8 + 1 = 4$ and invited mental solutions. Within a few seconds Antti offered *twenty-four* as the solution, to which Sami commented, *twenty-four. Let's try it.* While pointing at the board he added, *twenty-four divided by eight is three. And add one to it. Yes. Well, this was easy.* Finally, he demonstrated, through the use of his board sponge, a *covering up* method as discussed by Wijers (2001). Student input was restricted to answering closed arithmetical question, after which he commented that, *this works with some easy equations like this.*

5.2.2. Emese's activation phase

Emese began by posing oral problems like, *Kala is twice as old as her sister; the sum of their ages is 24, how old are they?* Each problem was solved individually before solutions were shared publicly. Next, the class was split into four groups with each given a word problem for translating into an equation. One group's problem was, *some friends went on a trip. The first day they covered just 2 km. The second day they covered 2/10 of the remaining journey. If they covered 6 km on the second day, how long was their journey?* After several minutes a member of each group explained how its equation had been derived before writing a symbolic form on the board. At this point it became clear that all groups had been given isomorphic problems as each representative wrote the same equation, $0.2(x - 2) = 6$. Lastly, a volunteer, exploiting a *thinking backwards* strategy, obtained a solution of 32. However, he had to have some support from Emese with regard to his managing division by 0.2. Lastly, Emese checked the solution against the text of each of the four problems.

5.2.3. Pauline's activation phase

Pauline, exploiting prepared acetates incorporating graphical representations of the balance scale, modelled an analytical solution to $x + 7 = 9$, which was then summarised symbolically before $x - 2 = 10$, $3x = 8$ and $x/3 = 7$ were managed in the same way. This was followed by her summarising the relationship between each of her four exemplars and their respective formalisations. In this respect, she had prepared on her acetate the following structural relationships

$$a = b \Rightarrow a + c = b + c$$

$$a = b \Rightarrow a - c = b - c$$

$$a = b \Rightarrow a.c = b.c$$

$$a = b \Rightarrow \frac{a}{c} = \frac{b}{c}.$$

The lesson ended with her setting a homework whereby solutions to equations like $x - 3 = 10$, $200 - x = 20$ were placed in a crossword grid. The following lesson answers were shared with particular attention being paid to $3/2x = 30$ and how division by $3/2$ was equivalent to multiplying by its inverse.

5.2.4. Summary

Summarising, Sami invited his students to solve intuitively two arithmetic equations, and used the latter to introduce a cover-up method that he never again mentioned. Emese exploited various word problems; initially to revisit the processes of *undoing* and latterly to derive arithmetic equations from *realistic* contexts and solve them with a thinking backwards strategy. Pauline privileged an explicit revision of arithmetical structures and their role in the solution of less straightforward arithmetic equations. In so doing, she made an explicit reference to the balance.

5.3. The exposition phase

All three teachers began their formal exposition by presenting their students with an algebraic equation, seemingly in the knowledge that intuitive methods would fail. The means by which the equation was presented varied, as did the manner by which it was solved, but all three introduced an algebraic equation which they knew, and made clear to their students, could not easily be solved by intuitive methods.

5.3.1. Sami's exposition phase

Sami, having invited two expressions, wrote $5x + 3 = 2x - 8$ and invited solutions. He waited patiently for some form of response that never came. He commented, with a smile, that *it's not that easy anymore. Is it? OK. This is a little bit more challenging and harder than before, especially as there are unknowns in both expressions. They are difficult to do mentally, and the covering up technique does not work either.* At this point Sami commented that *an equation is like a scales. In principle, if you have it in balance then the equation is true*, before stretching out his arms to demonstrate the effect of different actions, invited from the class, on the scales. Returning to the equation, he asked what could be subtracted from both sides of the equation. Someone suggested x and Sami, without comment, wrote $4x + 3 = x - 8$. Another volunteer suggested subtracting $2x$, at which point Sami wrote, with little student input:

$$\begin{array}{rcl} 5x + 3 = 2x - 8 & | & - 2x \\ 3x + 3 = -8 & | & - 3 \\ 3x = -11 & | & \end{array}$$

After some student uncertainty with regard to the next step, Sami, having asserted that they should divide by three as division is the opposite of multiplication, led the class to the solution $x = -11/3$, after which it was checked. Lastly, individual seatwork was set from a text book.

5.3.2. Emese's exposition phase

Emese began her second lesson by reading and writing simultaneously a word problem. It went, on two consecutive days the same weight of potatoes was delivered to the school's kitchen. On the first day 3 large bags and 2 bags of 10 kg were delivered. On the second day 2 large bags and 7 bags of 10 kg were delivered. If the weight of each large bag was the same, what was the weight of the large bag? A discussion ensued from which it emerged that an equation could be constructed and that x would represent the weight of the large bag. Shortly after this a volunteer wrote $3x + 20 = 2x + 70$ on the board. Next, having established by means of a series of question that intuitive strategies were now insufficient, Emese drew a picture of a scale balance with the various bags represented on both sides. Drawing on a student's suggestion she erased two small bags from each side, leaving a representation of $3x = 2x + 50$. Next she erased two large bags from each side to show one large bag balancing 5 small. Then, in response to her request, students volunteered sufficient for her to write alongside her drawings:

$$\begin{array}{rcl} 3x + 20 = 2x + 70 & | & - 20 \\ 3x = 2x + 50 & | & - 2x \\ x = 50 \text{ kg} & & \end{array}$$

Finally, Emese reminded her class of the importance of checking and did so.

5.3.3. Pauline's exposition phase

Midway through her second lesson, smiled and said, *and now, here comes the difficult equation*, before writing $6(x - 5) - 8 = x - 3$ on the board. There were some gasps from her students and she added, *yes? It seems a bit complicated, but you will see that it might have been worse and by the end of the lesson, you will be able to solve it perfectly.* She asked her students what they thought they should do first and Sara suggested removing the brackets. This initiated a process, lasting more than 20 min, whereby Pauline sought directions from her students at every stage and, very painstakingly, wrote the algebraic representation of the current step on the left and the justifications on the right. Space prevents a detailed account, although what follows represents a fragment of what was spoken and written.

Eliminate brackets

$$6(x - 5) - 8 = x - 3$$

At this point, Pauline drew from her students notions of associativity and commutativity before settling on distributivity as the warrant for what she was about to do. With her students' instructions she wrote the next line, drawing on a brief discussion around the product of 6 and -5 in relation to the rule of brackets, before concluding that there was the possibility of calculation. She wrote

Calculate if possible

$$6x - 30 - 8 = x - 3 \quad (2)$$

She continued to write the result of the most recent action on the left, as shown below.

Subtract x

$$6x - 38 = x - 3 \quad (3)$$

The discussion continued as follows:

Pauline What could I do next? Suppose I want all the terms in x in the left hand expression, so this x (pointing to the right hand side expression) needs to be removed. How can I remove this from the right hand expression? How can I remove this, what should I do with that? Okay. What might I do? *Mireille*

Subtract x .

Pauline Subtract x . Yes. Here we have $x - 3$. So if I subtract x , then x is removed from here to leave only -3 . So, I subtract x . Of course, if I subtract x from the right term, what should I do also? *Robert*?

Robert Subtract x from the left term.

Pauline Of course, subtract x from the left term too, otherwise it is no longer in balance.

This led to her writing, on the left hand side of the board,

$$5x - 38 = -3$$

The discussion continued in a similar fashion. Eventually the solution, $x = 7$, was obtained and, having discussed its uniqueness, Pauline undertook a check.

5.3.4. Summary

In summary, all three teachers presented their students with algebraic equations with, it seems, the intention of creating contexts whereby intuitive approaches could no longer be exploited. All three invoked the balance as an underlying principle and as the episodes played out, no evidence was seen of students failing to comprehend its significance as predicted by [Pirie and Martin \(1997\)](#). Indeed, Emese and Pauline exploited graphical representations of the balance in their treatments while Sami's acting of the balance is likely to have had a comparable effect. Interestingly, the extent to which the balance was sustained throughout the exposition varied. Sami, having introduced the balance, made little use of it during his rather directed exposition. Moreover, despite an implicit acceptance of his first student's subtraction of x , his subsequent actions indicated that he had a clear view as to what was acceptable. His solution was annotated conventionally although he invited no student input into its introduction. Emese exploited a realistic word problem to warrant the construction of her equation. She sustained the balance throughout her presentation, made explicit the relationship between her sketches and the symbolic representation and questioned her students constantly. Pauline offered the most complex of equations, deliberately provoking a frisson of excitement in her students. Her solution process, which was driven by many questions, was very formal and invoked a number of concepts studied earlier to highlight inter-topic and structural links. Both Sami and Pauline operated in exclusively mathematical worlds, while Pauline and Emese included checks at the conclusions of their expositions.

5.4. The consolidation phase

The fourth phase, lasting two or three lessons, provided various opportunities for students to consolidate earlier learning and further develop equations-related conceptual and procedural understandings.

5.4.1. Sami's consolidation phase

Throughout this period, which lasted three lessons, Sami used various forms of equation to consolidate his students' equation solving competence and highlight new insights. For example, at the start of his fourth lesson, he posed the following equation, $5(x - 1) = 3x + 7$, before initiating a public solution. A student, drawing on the previous lesson's work, volunteered that they should eliminate the brackets and the following was written, with no further student input, $5x - 5 = 3x + 7$. At this

point, Sami alerted his students to the fact that all their work so far had been leading up to an understanding that when an object changes sides of an equation its sign changes also. Thus, this insight was formalised thus

$$\begin{array}{ll} 5x - 5 = 3x + 7 & | \text{Change side change sign} \\ 5x - 3x = 7 + 5 & \\ 2x = 12 & | : 2 \\ x = 6 & \end{array}$$

Later problems, for example, $4x - (x + 3) = 3x - 3$, led to the collectively derived solution that $3x - 3 = 3x - 3$ and the understanding that any value of x satisfies the equation, while $6(2x - 2) = 3(4x + 1)$ led to the contradiction $-12 = 3$ and the realisation that a solution did not exist. The sequence of lessons closed with equations of the form $4x - 3(2x - 5) = 7(2x + 7)$.

5.4.2. Emese's consolidation phase

Emese spent the last three lessons of her sequence alternating equations drawn from the world of mathematics and *realistic* word problems. For example, a typical example of Emese's mathematical world equations was $15 - \{1 - 2[x - (3 - x)]\} = 72$, while a typical word problem was *a stake is driven through a pond into the ground. If 1/4 of the stake's length is in the ground, 3/5 is in water and 2.8 m is above the water, how long is the stake?* In such problems, which were clearly non-trivial, Emese ensured that students worked constantly with brackets, negatives and fractions. In every case, a problem was posed, a short period of individual working followed before a public discussion of the solution took place.

5.4.3. Pauline's consolidation phase

Pauline began her third lesson by going through the homework set the previous lesson. One of the problems, of only a few arithmetic equations students were asked to solve, was $4(x + 15) = 180$. The solution was discussed publicly with, in particular, Pauline reminding her students of her expectations with respect to annotations. The final solution appeared as below

$$\begin{array}{ll} 4(x + 15) = 180 & \text{Eliminate brackets} \\ 4x + 60 = 180 & \text{Get all terms in } x \text{ to one side and numbers to the other} \\ 4x = 180 - 60 & \text{Calculate where possible} \\ 4x = 120 & \text{Divide both sides by 4} \\ x = 30 & \end{array}$$

The complexity of the equations increased during the final two lessons, with the test, given at the end of the fifth lesson, comprising three problems. These were

$$5(p + 2) = 6 - 3; \quad (14 - 2x) - (x + 12) = x - 2; \quad \frac{-3}{4y} = \frac{-2}{3}.$$

5.4.4. Summary

In summary, all three teachers set increasingly complex exercises involving algebraic equations that incorporated brackets and both negative and fractional coefficients. Sami and Pauline located all their exercises within mathematics-only worlds, while Emese alternated between increasingly complicated algebraic and realistic word problem-derived equations. In different ways, all three teachers derived new insights from these tasks. Sami highlighted problematic special cases and introduced his preferred approach linked to the change the side and change the sign rule. Pauline formalised a set of procedures to which her students were expected to subscribe, while Emese, in always inviting and comparing multiple solution strategies, discussed notions of efficiency and elegance. Pauline included a test. The manner in which tasks were completed and solution shared varied with Sami and Pauline sharing solutions after several problems had been completed while Emese always shared solutions after each problem had been solved individually.

6. Discussion

In this paper we have presented the approaches to linear equations adopted by three, considered locally as effective, case study teachers, from Finland, Flanders and Hungary. In the following we compare their lesson sequences against the issues identified in the literature review. We acknowledge that such an examination is independent of the outcomes of their teaching but believe there are aspects of all three sequences likely to foster students' understanding not only of the nature of a linear equation but also the means by which it can be solved.

6.1. Examining the lesson sequences against the equations-related literature

Firstly, in encouraging their students to "understand that the expressions on both sides of the equals sign are of the same nature" (Filloy & Rojano, 1989, p. 19), all three teachers emphasised the structural properties of equations. They did this in

various ways, although their common emphases on algebraic equations as the provocations for their expositions and their invocations of the balance were clearly important in this regard. Sami and Emese, in the manner of Vlassis (2002), introduced the balance only when it became clear that students were unable to solve an algebraic equation by intuitive methods; while Pauline alluded to the balance during her activation, which involved the solution of arithmetic equations, before moving onto her formal algebraic equations-based exposition in much the same way as Warren and Cooper (2005).

Secondly, and mindful that a “relational view of the equal sign is necessary not only to meaningfully generate and interpret equations but also to meaningfully operate on equations” (Knuth et al., 2006, p. 309), the extent to which case study teachers’ explicitly addressed concerns relating to the equals sign varied. For example, the definitions presented by Sami and Emese presented a relational rather than operational interpretation of the equals sign. That is, by defining an equation as two expressions connected by an equals sign they avoided reinforcing the equals sign as an instruction to operate. Pauline’s approach was less unambiguous. Her Simpsons-derived equation highlighted the relational aspects of an algebraic equation, but did so implicitly rather than explicitly. All three exploited arithmetic equations during their activation phases, but, importantly, did so in ways that privileged their conceptual properties, which research has indicated is likely to be more successful than procedural interventions (Rittle-Johnson & Alibali, 1999). However, with regard to discussion, a key factor in overcoming students’ operational perspectives on the equals sign (Rittle-Johnson et al., 2011; Sáenz-Ludlow & Walgamuth, 1998), both Emese and Pauline engaged their students in extensive public discussion, whereas Sami tended to seek responses to closed question, typically focused on arithmetical matters, less likely to facilitate the necessary conceptual understanding.

Thirdly, and mindful of Pirie and Martin’s (1997) concerns over the balance and students’ management of negatives, all three teachers, particularly during consolidation phases, offered students frequent opportunities to work on algebraic equations involving not only negatives but fractions and brackets. Admittedly, evidence of the success of such approaches was limited to our observations of events, but little was seen to suggest that students were struggling with such matters. In particular, the frequent sharing of solutions observed in Emese’s lessons indicated that students were both confident and competent working with such entities. Moreover, not only did Sami and Pauline incorporate negatives into their initial expository equations but both invoked negatives and brackets throughout their consolidation phases, offering little evidence of students’ failure to manage successfully such issues.

Fourthly, for too many students, equation solving involves “simply moving symbols around” Noguera de Lima and Tall’s (2008, p. 10). Of the three lesson sequences, Emese’s approach, essentially alternating word problems with mathematical problems could be construed as addressing, at worst implicitly, such concerns. The ways in which each problem was discussed publicly provided constant opportunities for students to see that an equation was a representation of some form of reality, whether mathematical or realistic, and that the unknown was not an arbitrary symbol but had a meaning, a meaning of importance in the understanding of the particular equation. In similar vein, in addition to the meaning embedded in the Simpsons problem, Pauline’s detailed and highly structured exposition, when viewed alongside the public sharing of some, but not all, solutions also provided opportunities for equation solving to be imbued with meaning. However, her repeated invocation to *get all terms in x to one side and numbers to the other* may reflect an uncertain or ambivalent perspective on the goals of equation solving. In similar vein, Sami’s instruction, *change the side, change the sign*, presented an equally ambivalent perspective on the interaction of conceptual and procedural expectations.

Fifthly, with respect to word problems and the construction of equations, the three sequences present very different perspectives. At no point in his sequence of lessons did Sami offer any oral or text-based problem for translation into an equation; every problem he presented could be described as symbolically prepared. Pauline offered the single, Simpsons, problem, with all other being symbolically prepared equations around which her lessons played out. The significant difference lay with Emese, who alternated realistic word problems with symbolically prepared. Throughout Emese’s lessons, there was little discernible evidence of students failing to match key words with their corresponding symbols (Pawley et al., 2005) or experiencing syntactic and semantic difficulties (Mestre, 1988). Also, the word problems seen, whether Pauline’s single example or Emese’s many, were of such complexity that variable reversal was unlikely due, in most cases, to the unknown appearing on both sides of the equation.

Finally, all three teachers incorporated checks into their expositions, although Emese was the only one of the three who sustained checking throughout her sequence of lessons. Also, in contrast to research indicating not only that students find checking difficult (Perrenet & Wolters, 1994) but also that contemporaneous teaching of checking may be counter-productive (Pawley et al., 2005), the manner in which the solution to every one of Emese’s problems received public attention indicated that her students were confident in their understanding of both the need to check and the procedures for so doing.

6.2. Key similarities and differences

Looking beyond the equations-related literature, some other issues of interest emerged. The first concerned the three teachers’ exploitation of the balance. Despite the fact that all three introduced it, the ways in which it was systematically integrated and sustained varied considerably. The least consistent use was that of Pauline, who alluded to the balance during the preamble to her exposition but never referred to it again. Sami, as discussed above, acted out the balance as part of his exposition but, once he had completed the first example, never returned to it. Emese not only sustained the balance throughout her formal presentation but, unlike her Finnish and Flemish colleagues, made explicit the link between it and its symbolic representation of algebraic equations. Interestingly, Emese’s seemingly choreographed juxtaposition

of the balance, drawings and symbols resembled closely a Hungarian lesson observed ten years earlier by Andrews (2003), highlighting not only how Hungarian teachers appear to work within a unique didactic tradition (Anderson, Ryan, & Shapiro, 1989) but also how they seem to acknowledge Bruner's (1966) tripartite enactive, iconic and symbolic model of learning. Interestingly, and confounding Pirie and Martin's (1997) scepticism, not only did all students in all countries appear familiar with the balance but also, once it had been introduced, they seemed confident in their management of negatives, fractions and brackets.

A key similarity lay in the fact that all three teachers based their formal presentations on algebraic equations, which they knew could not be solved easily by intuitive methods. Indeed, Sami's long wait and *knowing smile* as his students struggled to solve $5x + 3 = 2x - 8$ highlighted well the need for an analytic approach. However, Pirie and Martin (1997, 161) describe equations teaching as a typically progressing "from simple equations of the form $x + b = d$ to $ax = d$ to $ax + b = d$ to $x - b = d$ to $ax - b = d$ before finally encountering $ax + b = cx + d$ ". Interestingly, no case study teachers adhered to such a sequence, mainly because their attention appeared focused on general forms of equation rather than, as seems to be the case in the English-speaking world, a sequence of activities that progress incrementally from a simple case through particular variations towards a general form. In this respect, Emese and Sami, seemed confident that teaching analytical approaches to equations that students could solve intuitively was a pointless activity.

The ways in which case study teachers used realistic problems highlights another interesting didactical issue. On the one hand, Sami used only problems located in a world of mathematics, with no word problems permeating his discourse. On the other hand, Emese alternated realistic with mathematical problems throughout her lesson sequence. Indeed, the potato problem, around which she constructed her exposition, was clearly realistic in its presenting an imaginably real (Van den Heuvel-Panhuizen, 2003) context from which an equation emerged. Somewhere between the two, Pauline exploited a single realistic problem to initiate her lesson sequence. Sadly, we argue, having derived an algebraic equation linking the sum of the children's ages to their mother's, she never returned to it. Consequently, an opportunity to link the formal treatment of algebraic equations to the introductory activity went unexploited. All other problems in Pauline's sequence were located in a mathematical world. Importantly, and by way of further confounding Pirie and Martin's (1997) scepticism, none of the realistic contexts insulted the intelligence of their students. Indeed, Pauline's introduction to the Simpsons problem yielded smiles of amusement and a tangible frisson of excitement. Moreover, our inference from repeated viewings of the lessons is that these students were not only comfortable with realistic problems but unproblematically understood that they were not to be construed as genuine attempts to model the real world; they were didactic tools developed to facilitate students' access to mathematical concepts and procedures.

In closing, we suggest that there is much in the three sequences above to challenge those with an interest in the teaching of linear equations. All three sequences offered insights into how teachers, defined locally as effective, addressed those problems identified in the literature as barriers to students' equation solving competence. In different ways, all three sequences, but particularly Emese's, confounded Pirie and Martin's (1997) well-known criticisms of the balance, while two, Pauline's and Emese's also confounded criticisms of realistic problems as a means to engage students and scaffold learning.

Acknowledgements

The authors acknowledge the generous support of the European Union, Socrates Action 6.1 Programme Code 2002-5048.

The METE project team was Erik De Corte, Fien Depaepe, Peter Op't Eynde and Lieven Verschaffel, The Catholic University of Leuven, Belgium; Paul Andrews, Gillian Hatch (now sadly deceased) and Judy Sayers, Cambridge, Manchester Metropolitan and Northampton Universities respectively, England; George Malaty and Tuomas Sorvali, University of Joensuu, Finland, Kati Fried, Sári Pálfalvi, Éva Szeredi and Judit Török, Eötvös Loránd University, Budapest, Hungary; and José Carrillo, Nuria Climent and Cinta Muñoz, University of Huelva, Spain.

References

- Alibali, M. (1999). How children change their minds: Strategy change can be gradual or abrupt. *Developmental Psychology*, 35(1), 127–145.
- Alibali, M., Knuth, E., Hattikudur, S., McNeil, N., & Stephens, A. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning*, 9(3), 221–247.
- Anderson, L., Ryan, D., & Shapiro, B. (1989). *The IEA classroom environment study*. Oxford: Pergamon.
- Andrews, P. (2003). Opportunities to learn in the Budapest mathematics classroom. *International Journal of Science and Mathematics Education*, 1(2), 201–225.
- Araya, R., Calfucura, P., Jiménez, A., Aguirre, C., Palavicino, A., Lacourly, N., et al. (2010). The effect of analogies on learning to solve algebraic equations. *Pedagogies: An International Journal*, 5(3), 216–232.
- Asquith, P., Stephens, A., Knuth, E., & Alibali, M. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable. *Mathematical Thinking and Learning*, 9(3), 249–272.
- Baroody, A., & Ginsburg, H. (1983). The effects of instruction on children's understanding of the 'equals' sign. *The Elementary School Journal*, 84(2), 199–212.
- Boulton-Lewis, G., Cooper, T., Atweh, B., Pillay, H., Wilson, L., & Mutch, S. (1997). Processing load and the use of concrete representations and strategies for solving linear equations. *Journal of Mathematical Behavior*, 16(4), 379–397.
- Brown, T., Eade, F., & Wilson, D. (1999). Semantic innovation: Arithmetical and algebraic metaphors written within narratives of learning. *Educational Studies in Mathematics*, 40(1), 53–70.
- Bruner, J. (1966). *Towards a theory of instruction*. Cambridge, MA: Harvard University Press.
- Capraro, M. M., & Joffrion, H. (2006). Algebraic equations: Can middle-school students meaningfully translate from words to mathematical symbols? *Reading Psychology*, 27(2–3), 147–164.
- Clarke, D. (2006). The LPS research design. In D. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* (pp. 15–36). Rotterdam: Sense.

- Da Rocha Falcão, J. (1995). A case study of algebraic scaffolding: From balance scale to algebraic notation. In L. Meira, & D. Carraher (Eds.), *19th international conference for the psychology of mathematics education* (pp. 66–73). Recife, Brazil: Universidade Federal de Pernambuco.
- Dickinson, P., & Eade, F. (2004). Using the number line to investigate the solving of linear equations. *For the Learning of Mathematics*, 24(2), 41–47.
- English, L., & Sharry, P. (1996). Analogical reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics*, 30(2), 135–157.
- Falkner, K., Levi, L., & Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Mathematics Teaching in the Middle School*, 6(4), 232–236.
- Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, 9(2), 19–25.
- Filloy, E., & Sutherland, R. (1996). Designing curricula for teaching and learning algebra. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 139–160). Dordrecht: Kluwer.
- Fong, H., & Chong, T. (1995). Solving algebraic word problems. *Mathematics Teaching*, 151, 34–35.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel.
- Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27(1), 59–78.
- Humberstone, J., & Reeve, R. (2008). Profiles of algebraic competence. *Learning and Instruction*, 18(4), 354–367.
- Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.
- Kieran, C. (2006). Research on the teaching and learning of algebra. In A. Gutiérrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 11–49). Rotterdam: Sense.
- Knuth, E., Stephens, A., McNeil, N., & Alibali, M. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, 37(4), 297–312.
- Liamputtong, P. (2010). *Performing qualitative cross-cultural research*. Cambridge: Cambridge University Press.
- Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in 'filling' the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics*, 39(1), 131–147.
- MacGregor, M., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education*, 24(3), 217–232.
- MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11–15. *Educational Studies in Mathematics*, 33(1), 1–19.
- McNeil, N., & Alibali, M. (2005). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development*, 6(2), 285–306.
- Mestre, J. (1988). The role of language comprehension in mathematics and problem solving. In R. Cocking, & J. Mestre (Eds.), *Linguistic and cultural influences on learning mathematics* (pp. 201–220). Hillsdale, NJ: Lawrence Erlbaum.
- Nogueira de Lima, R., & Tall, D. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics*, 67(3), 3–18.
- Pawley, D., Ayres, P., Cooper, M., & Sweller, J. (2005). Translating words into equations: A cognitive load theory approach. *Educational Psychology*, 25(1), 75–97.
- Perrenet, J., & Wolters, M. (1994). The art of checking: a case study of students' erroneous checking behavior in introductory algebra. *Journal of Mathematical Behavior*, 13(3), 335–358.
- Pirie, S., & Martin, L. (1997). The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations. *Educational Studies in Mathematics*, 34(2), 159–181.
- Rittle-Johnson, B., & Alibali, M. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, 91(1), 175–189.
- Rittle-Johnson, B., Matthews, P., Taylor, R., & McEldoon, K. (2011). Assessing knowledge of mathematical equivalence: A construct-modeling approach. *Journal of Educational Psychology*, 103(1), 85–104.
- Sáenz-Ludlow, A., & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol. *Educational Studies in Mathematics*, 35(2), 153–187.
- Sahlberg, P. (2011). The professional educator: Lessons from Finland. *American Educator (Summer)*, 34–39.
- Schmidt, W., McKnight, C., Houang, R., Wang, H., Wiley, D., Cogan, L., et al. (2001). *Why schools matter: A cross-national comparison of curriculum and learning*. San Francisco: Jossey-Bass.
- Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior*, 14(1), 15–39.
- Sherman, J., & Bisanz, J. (2009). Equivalence in symbolic and nonsymbolic contexts: Benefits of solving problems with manipulatives. *Journal of Educational Psychology*, 101(1), 88–100.
- Stacey, K., & MacGregor, M. (2000). Learning the algebraic method of solving problems. *The Journal of Mathematical Behavior*, 18(2), 149–167.
- Steinberg, R., Sleeman, D., & Ktorza, D. (1991). Algebra students' knowledge of equivalence of equations. *Journal for Research in Mathematics Education*, 22(2), 112–121.
- Swafford, J., & Langrall, C. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education*, 31(1), 89–112.
- Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, 54(1), 9–35.
- Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, 49(3), 341–359.
- Warren, E., & Cooper, T. J. (2005). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal*, 17(1), 58–72.
- Wijers, M. (2001). *How to deal with algebraic skills in Realistic Mathematics Education* Paper presented at the 12th ICMI study conference: The future of the teaching and learning of algebra, University of Melbourne, December 9–14, 2001.
- Wollman, W. (1983). Determining the sources of error in a translation from sentence to equation. *Journal for Research in Mathematics Education*, 14(3), 169–181.